Sampling the Laplace Transform

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Abstract

I would like to emphasize the connection between the Laplace and Fourier transforms in this document. In particular, we consider how the Laplace transform is a Fourier transform *preceded by* exponential regularization, in both the discrete and continuous-time cases. Then, the relationship between the two is shown in terms of sampling, in the same fashion as the Fourier transforms.

The goal of this document is not to dig into all of the details, integral expression, and general calculus of signals and transforms. Rather, the goal is to put the "algebra" of these transforms front-and-center. Because of this, statements will be made with little regard for conjugation, time-reversals, or constant factors. The reader should figure out how these arise through their own studies.

1 Exponentially regularized transforms

1.1 Recalling Fourier

Let us remember the "Four Fourier Transforms." The Fourier transform and Fourier series is described in both continuous time and in discrete time, transforming functions in the following ways:

$$
\begin{aligned}
(\mathbb{R} \to \mathbb{C}) &\xleftrightarrow{FT} (\mathbb{R} \to \mathbb{C}) \\
(S^1 \to \mathbb{C}) &\xleftrightarrow{FS} (\mathbb{Z} \to \mathbb{C}) \\
(\mathbb{Z} \to \mathbb{C}) &\xleftrightarrow{FT} (S^1 \to \mathbb{C}) \\
(S^1_N \to \mathbb{C}) &\xleftrightarrow{FS} (S^1_N \to \mathbb{C}).
\end{aligned}
$$

The continuous-time Fourier transform (CTFT), for instance, maps complex-valued functions on the real line, to complex-valued functions on the real line. Notationally, this is denoted $(\mathbb{R} \to \mathbb{C}) \xleftrightarrow{FT} (\mathbb{R} \to \mathbb{C})$. Similarly, the discrete-time Fourier transform (DTFT) maps complex-valued functions on the integers to complex-valued functions on the circle: $(\mathbb{Z} \to \mathbb{C}) \xleftrightarrow{FT} (S^1 \to \mathbb{C})$.

We claimed that our emphasis is on the algebraic aspects of signal transformations, at the expense of the analytic/calculative aspects. We will deviate from that briefly to motivate the Laplace and z-transforms. For the CTFT and DTFT, the Fourier transform is at its best when applied to *absolutely integrable/summable signals.* Indeed, a signal being absolutely integrable/summable allows for the signal to "periodized" in a well-defined way, allowing us to derive the CTFT and DTFT from the CTFS and DTFS, respectively.

Not all signals of interest are absolutely integrable. A very simple one is the unit step:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

The unit step is definitely not integrable! This is due to the fact that it does not decay to zero as $t \to \infty$; of course, the unit step (very quickly!) attains zero as $t \to -\infty$. The only issue, then, is the heavy tail of the function for large, positive values of t. The method of the Laplace transform, then, is to regularize these tails in a *bijective fashion* in order to achieve integrability.

1.2 Regularized transforms

One class of functions that decays very quickly is the class of real exponentials. For $\sigma > 0$, the function $e^{-\sigma t}$ approaches zero as $t \to \infty$ faster than any rational function (such as, $1/p(t)$ for some polynomial p). Because of this, for any $\sigma > 0$, the signal $u(t)e^{-\sigma t}$ is absolutely integrable, and thus has a well-defined CTFT. Moreover, we can recover $u(t)$ from $u(t)e^{-\sigma t}$, because $e^{-\sigma t} \neq 0$ for all t.

This multiplication by $e^{-\sigma t}$ constitutes a sort of *lifting* parameterized by $\sigma \in \mathbb{R}$. We might think of the diagram in this way:

$$(\mathbb{R}_t \to \mathbb{C}) \overset{\mathsf{LIFT}}{\longleftrightarrow} (\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}) \overset{FT}{\longleftrightarrow} (\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C}).$$

How should we read this? First, focus on the rightmost arrow relation, since that is the most familiar. The set denoted $(\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C})$ describes "lifted signals," which are functions of both $t, \sigma \in \mathbb{R}$. We find it useful to distinguish the two, saying that $t \in \mathbb{R}_t$ and $\sigma \in \mathbb{R}_\sigma$. The Fourier transform, then, is taken *only with respect to the t variable*, yielding a family of functions in the frequency domain that also depend on σ, denoted $(\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C})$.

Of course, the functions in $(\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C})$ are not arbitrary: they are given by exponentials involving σ being multiplied by signal on $\mathbb{R}$. We denote this multiplication by LIFT, illustrated in Fig. 1.

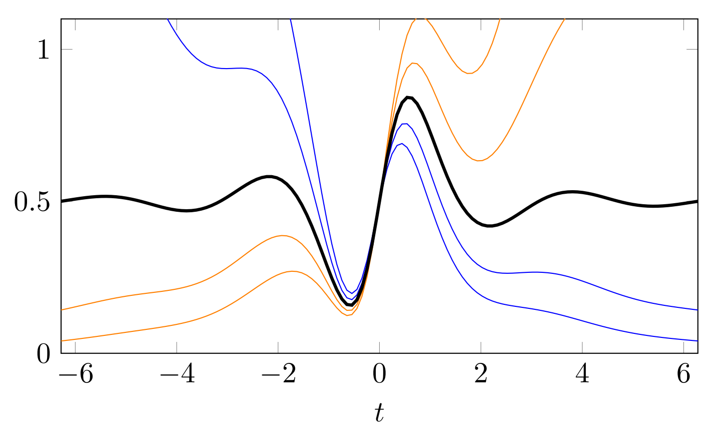

Figure 1: A signal $f : \mathbb{R}_t \to \mathbb{C}$ (black) and its lifted version $\mathsf{LIFT}[f] : \mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}$. Each orange curve corresponds to a distinct value of $\sigma < 0$, and each blue curve corresponds to a distinct value of $\sigma > 0$.

Looking back to the unit step function from before, we chose an arbitrary $\sigma > 0$ and took the product $u(t)e^{-\sigma t}$. What LIFT does is compute this product *for all values of* $\sigma \in \mathbb{R}$.

Exercise 1: Injectivity of LIFT

Convince yourself that for any distinct signals $x, y : \mathbb{R}_t \to \mathbb{C}$, their lifted versions are distinct.

We can also define a notion of LIFT for the Fourier domain, that is, a map whose diagram looks like

$$(\mathbb{R}_\omega \to \mathbb{C}) \overset{\mathsf{LIFT}}{\longleftrightarrow} (\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C}).$$

To define this in a nice analytical form, we would have to define the Fourier transform of an exponential e^{-st} and then convolve a function in the Fourier domain with it: that would require too much work, so we will define LIFT in this setting to be such that it commutes with the Fourier transform applied to time-domain signals. That is to say, we have the following commutative diagram:

$$\begin{array}{ccc}
(\mathbb{R}_t \to \mathbb{C}) & \longleftarrow \mathsf{LIFT} \longrightarrow & (\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}) \\
\uparrow & & \uparrow \\
\text{CTFT} & & \text{CTFT} \\
\downarrow & & \downarrow \\
(\mathbb{R}_\omega \to \mathbb{C}) & \longleftarrow \mathsf{LIFT} \longrightarrow & (\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C})
\end{array}$$

Of course, a lifted signal need not be absolutely integrable for all values of σ. The Fourier transform, then, is only to be applied for values of σ for which the lifted signal is absolutely integrable. We won't worry too much about this, though: this is why we describe a Laplace transform by including its region of convergence. Before moving on, let us state what the Laplace transform is at a high level.

Definition 1: The Laplace transform

The *Laplace Transform* is described by the diagram

$$(\mathbb{R}_t \to \mathbb{C}) \overset{FT \circ \mathsf{LIFT}}{\longleftrightarrow} (\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C}).$$

That is, the Laplace transform maps signals $(\mathbb{R}_t \to \mathbb{C})$ to functions $(\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C})$.

We see from the description of the Laplace transform as a composition of LIFT and FT that the part of the domain $\mathbb{R}_\sigma$ is introduced by LIFT, and the part of the domain $\mathbb{R}_\omega$ is the dual to the time-domain $\mathbb{R}_t$ transformed via FT.

Let's apply a similar method to DT signals. Rather then multiplying by CT exponentials e^{-st}, we will multiply DT signals by DT exponentials r^{-n}, for positive real numbers r. We ignore $r = 0$, since that is trivial. For instance, the DT unit step is

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

The DT unit step is not absolutely summable, and thus does not have a well-defined Fourier transform. However, if we choose $r > 1$, the DT signal $u[n]r^{-n}$ is absolutely summable, and thus has a well-defined Fourier transform. Once again, this multiplication by r^{-n} constitutes a *lifting* parameterized by $r \in \mathbb{R}^{>0}$. We will draw the new diagram in this way:

$$(\mathbb{Z} \to \mathbb{C}) \overset{\mathsf{LIFT}}{\longleftrightarrow} (\mathbb{R}^{>0} \times \mathbb{Z} \to \mathbb{C}) \overset{FT}{\longleftrightarrow} (\mathbb{R}^{>0} \times S^1 \to \mathbb{C}).$$

Notice the change in domain: because the DTFT has the form $(\mathbb{Z} \to \mathbb{C}) \longleftrightarrow (S^1 \to \mathbb{C})$, this is carried over to our description of the regularized transform in the DT case. In this diagram, the DTFT is taken with respect to the $\mathbb{Z}$ variable in the lifted space. This is what we call the z-transform, which we describe below.

Definition 2: The z-transform

The *z-transform* is described by the diagram

$$(\mathbb{Z} \to \mathbb{C}) \overset{FT \circ \mathsf{LIFT}}{\longleftrightarrow} (\mathbb{R}^{>0} \times S^1 \to \mathbb{C}).$$

That is, the z-transform maps signals $(\mathbb{Z} \to \mathbb{C})$ to functions $(\mathbb{R}^{>0} \times S^1 \to \mathbb{C})$.

The same concerns about convergence for different values of $r > 0$ hold as in the CT case, and we will discard those concerns in the same way.

2 Sampling and approximation of regularized transforms

For both the Laplace transform and the z-transform, we have described it by a composition of the form $FT \circ \mathsf{LIFT}$. Let us shorten this to $LT = FT \circ \mathsf{LIFT}$, just to give it an easy name. So, we have the two statements:

$$(\mathbb{R}_t \to \mathbb{C}) \overset{LT}{\longleftrightarrow} (\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C})$$
$$(\mathbb{Z} \to \mathbb{C}) \overset{LT}{\longleftrightarrow} (\mathbb{R}^{>0} \times S^1 \to \mathbb{C}).$$

Neat! How are the two related? Recall our notion of an *approximation map*. For example, the approximation map corresponding to sampling CT signals is denoted

$$(\mathbb{R}_t \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C}).$$

The corresponding approximation map in the Fourier domain describes "periodization," which is convolution with a pulse train:

$$(\mathbb{R}_\omega \to \mathbb{C}) \rightsquigarrow (S^1 \to \mathbb{C}).$$

The key to our understanding comes from the following commutative diagram for the Fourier transform:

$$\begin{array}{ccc}
(\mathbb{R}_t \to \mathbb{C}) & \longleftarrow \text{CTFT} \longrightarrow & (\mathbb{R}_\omega \to \mathbb{C}) \\
\big\downarrow\wr & & \big\downarrow\wr \\
(\mathbb{Z} \to \mathbb{C}) & \longleftarrow \text{DTFT} \longrightarrow & (S^1 \to \mathbb{C})
\end{array}$$

Do we have something similar for the Laplace transform? Of course! If we can build suitable approximation maps for the lifted signal and Fourier spaces, we get a commutative diagram for free. What we need are maps of the following type:

$$(\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}) \rightsquigarrow (\mathbb{R}^{\geq 0} \times \mathbb{Z} \to \mathbb{C})$$
$$(\mathbb{R}_\sigma \times \mathbb{R}_\omega \to \mathbb{C}) \rightsquigarrow (\mathbb{R}^{\geq 0} \times S^1 \to \mathbb{C}).$$

This is not too hard: the approximation map for sampling CT signals looks like

$$\begin{aligned}(\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}) &\rightsquigarrow (\mathbb{R}^{\geq 0} \times \mathbb{Z} \to \mathbb{C}) \\ x(\sigma, t) &\mapsto x[r, n] := x(\log r, nT).\end{aligned}$$

The approximation map in the Fourier domain is defined in a similar way: sending σ to e^σ and taking the periodization of the Fourier transform after multiplying by the relevant exponential.

Notably, these approximation maps are fully compatible with LIFT, in the sense that the sequence of operations

$$(\mathbb{R}_t \to \mathbb{C}) \xleftrightarrow{\mathsf{LIFT}} (\mathbb{R}_\sigma \times \mathbb{R}_t \to \mathbb{C}) \rightsquigarrow (\mathbb{R}^{\geq 0} \times \mathbb{Z} \to \mathbb{C})$$

is equivalent to

$$(\mathbb{R}_t \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C}) \xleftrightarrow{\mathsf{LIFT}} (\mathbb{R}^{\geq 0} \times \mathbb{Z} \to \mathbb{C}).$$

A similar equivalence holds for the approximation and lifting maps in the Fourier domain.

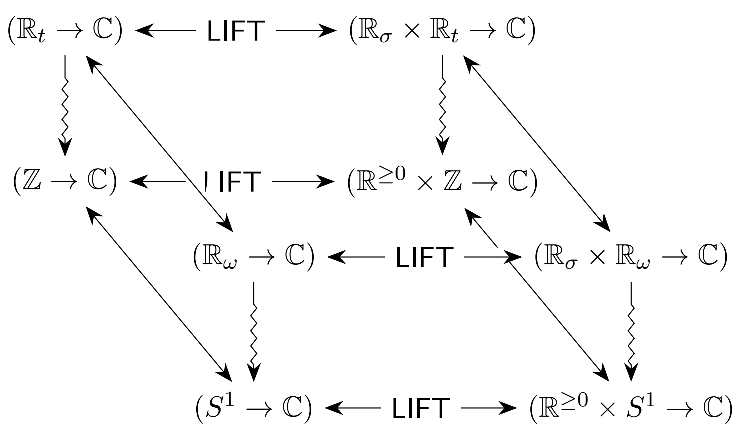

Figure 2: Laplace chart: diagonal lines indicate Fourier transforms, and squiggly lines indicate approximation maps.

With all of these constructions, we can glue everything that we have learned together into one big diagram, shown in Fig. 2. This figure is a bit shocking at first, and could perhaps use some color or a more friendly font to make it less intimidating. We can learn to read it one step at a time.

First, notice that all of the corners in the "back" of the diagram correspond to signals in time and their lifted counter parts. The upper corners correspond to CT signals, and the lower corners correspond to DT signals. The corners in the "front" of the diagram are simply the Fourier transform of those signals in the back: hence, the upper corners have a real-valued frequency variable, while the lower corners have a circular frequency variable, corresponding to the CTFT and DTFT, respectively.

The squiqqly arrows denote the approximation maps in time and frequency. To go from CT to DT, you just take uniform samples. The approximation map in the Fourier domain is a bit more complicated, as discussed above.

3 Properties of the Laplace/z-transforms

In any sufficiently thorough study of the Laplace and z-transforms, there will be a table of properties discussing the properties of the region of convergence (ROC) of these transforms. If you are lucky, it might even be explained to you that they have something to do with one another. Let us go through a few of the usual properties, and see how they fit in to our discussion of lifting and sampling.

The first property of the ROC of the Laplace/z-transform is the following:

Property 1: Shape of the ROC

The ROC of $X(s)$ consists of vertical strips in the s-plane. Similarly, the ROC of $X(z)$ consists of rings centered about the origin (annuli) in the z-plane.

These two statements give us an idea of what diagrams showing the ROC usually look like. If we think of DT signals $x[n]$ as being approximations of CT signals $x(t)$ that are related by a sampling map, we know that it is natural to associate their Fourier transforms via a sampling map, too (where sampling in the Fourier domain is done by periodization). As Fig. 2 tells us, the lifted version of the Fourier transform also yields a relation between the s-plane and the z-plane.

Paying attention only to how LIFT transforms the $\mathbb{R}_\sigma$ variable, the transformation is given by $\sigma \mapsto e^\sigma$. That is to say, we can interpret the z-plane as the exponential of the s-plane, at least as far as the real axis is concerned. Since the imaginary axis in the s-plane corresponds to the frequency variable of the CTFT, our transformation between the s-plane and z-plane should resemble the periodization approximation map. It fortunately does: periodization of a frequency parameter ω essentially *wraps ω around the unit circle*, which is does via the map sending ω to $e^{j\omega}$.

Noting that the s variable in the Laplace transform is typically decomposed as $s = \sigma + j\omega$, we see that $e^s = e^\sigma \cdot e^{j\omega}$. The first term in the product is the real part of z (the magnitude), while the second part is the frequency variable of the DTFT.

Let us return to our property. Take the property of the ROC in the s-plane consisting of vertical strips as a given.

Exercise 2: Exponential of a vertical strip

Let V be a vertical strip in the s-plane. What is e^V shaped like in the z-plane?

I leave the derivations for the other properties of the Laplace and z-transforms as an exercise.

4 The lesson, and some takeaways

The Laplace and z-transform are very useful tools in the analysis of LTI systems, and are worthy objects of study on account of that alone. However, their use in analyzing signals that are difficult to grok using bare Fourier analysis teaches an important lesson. Returning to the example of the unit step: since it is not integrable, the Fourier transform does not exist using the *direct definition*. That is, the following integral does not converge in the usual sense:

$$\widehat{u}(\omega) = \int_0^\infty e^{-j\omega t} dt.$$

Instead of throwing our hands up in the air, we can *indirectly* perform the Fourier transform, by sneaking in a regularizing term. In the case of the Laplace and z-transforms, this regularizing term is an exponential with suitable decay properties to ensure integrability. The more poorly-behaved the signal being analyzed is, the more regularization is required.

This approach can be used to make some bold claims, as long as you are willing to place qualifiers on those claims. I leave the following exercise based on this idea:

Exercise 3: CTFT of the unit step

Take the Laplace transform of the unit step $u(t)$. By taking $\sigma \to 0^+$, argue that the Fourier transform of the unit step is given by $\widehat{u}(\omega) = \frac{1}{j\omega}$, ignoring the singularity at $\omega = 0$.

Note that this claim agrees with how we usually think of the Fourier transform of the unit step by using the convolution theorem, even though the Fourier transform as defined directly doesn't necessarily exist. By using *regularization*, we can indirectly define these behaviors, which usually hold when applied in sufficiently nice situations.