Fourier series decay of Hölder-continuous functions on the n-torus

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14 March 2023

Denote the n-torus by $\mathbb{T}^n = \prod_{j=1}^n S^1$, where S^1 denotes the circle. We identify the circle with the half-open interval $[0, 2\pi)$, where the endpoints are identified with one another. This is the usual setup for considering periodic functions in Euclidean space – a periodic function on the real line, for example, is treated as a function on the circle.

For a function $f \in L^1(\mathbb{T}^n)$, we define the Fourier series of f to be the expansion

$$S[f](x) = \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) \cdot \exp\left(i\langle \xi, x\rangle\right),$$

where $i = \sqrt{-1}$, and each Fourier series coefficient $\hat{f}(\xi)$ is calculated as

$$\hat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{T}^n} f\left(x\right) \exp\left(-i\langle \xi, x\rangle\right) dx.$$

All of the usual justifications show that the Fourier series representation of f is indeed convergent to f in $L^1(\mathbb{T}^n)$.

1 Hölder continuity

We are of course not generally confronted with arbitrary square-integrable functions, but with functions that have greater regularity properties. Let $L \geq 0, p \geq 1$ be given. A function $f : \mathbb{T}^n \to \mathbb{R}$ is said to be Lipschitz with respect to the p-norm if there exists a constant $L \geq 0$ such that for all $x, y \in \mathbb{T}^n$,

$$\left|f(x) - f(y)\right| \leq L \cdot \|x - y\|_p.$$

Furthermore, if some $0 \leq \alpha \leq 1$ is given, f is said to be α-Hölder continuous with respect to the p-norm if there exists a constant $L \geq 0$ such that for all $x, y \in \mathbb{T}^n$,

$$\left|f(x) - f(y)\right| \leq L \cdot \|x - y\|_p^{\alpha}.$$

Observe that taking $\alpha = 1$ recovers the notion of a Lipschitz function, and that the space of α-Hölder continuous functions grows as $\alpha \to 0$. When $\alpha = 0$, the above condition implies that f is uniformly continuous (and given that the domain of f is compact, this just says that f is continuous). In all cases, the p-norm is to be understood in terms of the angular distances on the component circles making up to n-torus. Note that all Hölder continuous function are continuous, and are thus contained in $L^1(\mathbb{T}^n)$.

2 Integral modulus of continuity

For $f \in L^1(\mathbb{T}^n)$ and $h \in \mathbb{R}^n$, define the integral modulus of continuity as the quantity

$$\Omega(f,h) = \|f(t+h) - f(t)\|_{L^1}.$$

Again, $f(t+h)$ is meant to be understood via the periodic extension (angular addition). We now provide an extension of the theorem from [1, pp. 26]. First, a proposition is in order.

Proposition 2.1. *Let $\xi \in \mathbb{Z}^n$ be given such that $\xi \neq 0$. Suppose that $\eta \in \mathbb{R}^n$ is such that $\langle \xi, \eta \rangle = -1$. For $f \in L^1(\mathbb{T}^n)$, it holds that*

$$\left| \hat{f}\left(\xi\right) \right| \leq \frac{(2\pi)^{-n}}{2} \Omega\left(f, \pi\eta\right).$$

Proof. The proof is by direct calculation.

$$\begin{aligned}
\hat{f}\left(\xi\right) &= (2\pi)^{-n} \int_{\mathbb{T}^n} f\left(x\right) \exp\left(-i\langle \xi, x\rangle\right) dx \\
&= -\left(2\pi\right)^{-n} \int_{\mathbb{T}^n} f\left(x\right) \exp\left(-i\langle \xi, x - \pi\eta\rangle\right) dx \\
&= -\left(2\pi\right)^{-n} \int_{\mathbb{T}^n} f\left(x + \pi\eta\right) \exp\left(-i\langle \xi, x\rangle\right) dx.
\end{aligned}$$

Then, we observe the inequality

$$\begin{aligned}
\left| \hat{f}\left(\xi\right) \right| &= \frac{1}{2} \left| \hat{f}\left(\xi\right) + \hat{f}\left(\xi\right) \right| \\
&= \frac{(2\pi)^{-n}}{2} \left| \int_{\mathbb{T}^n} \left(f\left(x + \pi\eta\right) - f\left(x\right)\right) \exp\left(-i\langle \xi, x\rangle\right) dx \right| \\
&\leq \frac{(2\pi)^{-n}}{2} \Omega\left(f, \pi\eta\right),
\end{aligned}$$

completing the proof. □

This can then be applied to Hölder continuous functions by considering how the continuity conditions affect the integral modulus of continuity.

Theorem 2.2. *Let $\xi \in \mathbb{Z}^n, 0 \leq \alpha \leq 1, p \geq 1$ be given such that $\xi \neq 0$. If f is α-Hölder continuous on $\mathbb{T}^n$ with respect to the p-norm, then*

$$\left|\hat{f}(\xi)\right| \leq C \cdot \|\xi\|_q^{-\alpha},$$

where q is the Hölder dual[1] of p, and $C \geq 0$ is a constant independent of ξ.

Proof. We will prove this by selecting an $\eta \in \mathbb{R}^n$ such that $\langle \xi, \eta \rangle = -1$ and $\Omega(f, \pi\eta)$ is minimized. First, observe by assumption that

$$\Omega(f, \pi\eta) = \int_{\mathbb{T}^n} |f(x + \pi\eta) - f(x)| dx \leq \int_{\mathbb{T}^n} L\|\pi\eta\|_p^\alpha dx = (2\pi)^n \pi^\alpha L \|\eta\|_p^\alpha,$$

for some $L \geq 0$ dependent on f. So, we seek to minimize $\|\eta\|_p^\alpha$ subject to $\langle \xi, \eta \rangle = -1$. One can check, by properties of the dual of the p-norm for finite-dimensional spaces, that the minimum such value is attained as $\|\eta\|_p^\alpha = \|\xi\|_q^{-\alpha}$. Applying Proposition 2.1 yields the desired bound. □

Theorem 2.2 indicates that as $\alpha \to 1$ (that is, as the demands of the continuity condition strengthen up to the point of being Lipschitz), the guaranteed decay of the Fourier series becomes stronger. This conforms to intuition: smooth functions are low-pass. Moreover, in the proof, we see that the constant C is proportional to the bounding constant of the α-Hölder continuity condition.

3 Equivalence of norms

In Theorem 2.2, we went through the trouble of considering the necessary results for norms other than $p = 2$. When n is fixed, all p-norms are equivalent for $p \geq 1$, so the decay properties all say essentially the same thing, with some adjustment of constant factors. However, one may be interested in understanding a sequence of functions on toruses[2] of increasing dimension, in which case distinguishing between different values of p for the "sequential" regime may be useful. There is no need to go into the details in this note, though.

References

[1] Y. Katznelson, *An introduction to harmonic analysis*. Cambridge University Press, 2004.

[1] That is, the number q such that $p^{-1} + q^{-1} = 1$. When $p = 1$, the Hölder dual is given by $q = \infty$, and vice-versa.

[2] I do not like using Latin plural forms when I do not have to. If you are reading this aloud for some reason, feel free to say 'tori' if it makes you feel better.